$$\rho C_p \frac{\partial T}{\partial t} = k \nabla^2 T$$

- LUMPED $T(t)$
  - $\rho \forall C \frac{dT}{dt} = q_{out}$
  - (conv, rad.)
  - (insolation)
  - (generation)
- $\rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial n^2}$ (1D)
  - $T(t,x)$ or $T(t,r)$
    - Cart. $T(t,x)$
    - Cyl $T(t,r)$
    - Sph $T(t,\rho)$
  - $T(t,x,y,z)$, $T(t,r,\theta)$, $T(t,\rho,\varphi,\theta)$

I.C.

@ $t=0$

$T=T_i$

B.C.

@ $x=0$

$\frac{\partial T}{\partial x}=0$

By sym.

@ $x = \text{"}L\text{"} \rightarrow \infty$

Similarity transform

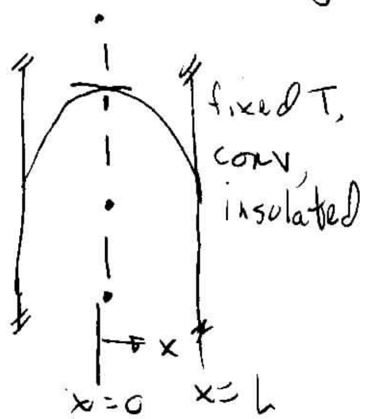
$\eta = \frac{x}{\sqrt{4\alpha t}}$

∞ solid

semi-∞ solid

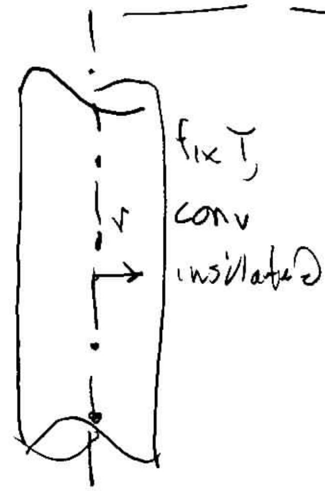
Transient 1-D analysis... $T(t,x)$ or $T(t,r)$ or $T(t,\rho)$

Comments about geometry ... (finite dimension)



symetric BC

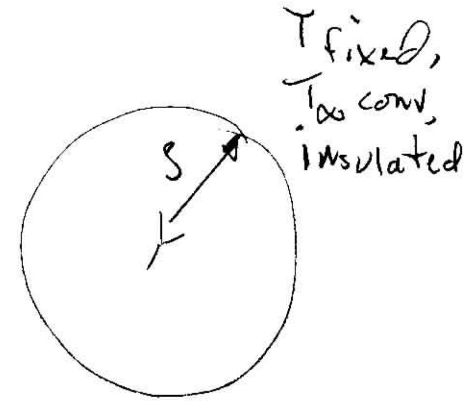
$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$



$T|_{r=0}$ finite

or

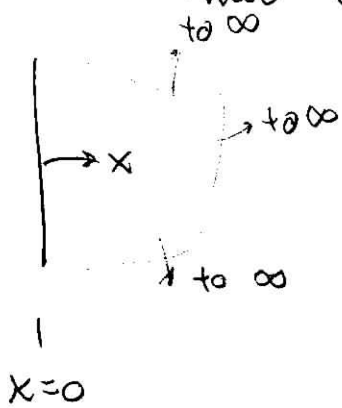
$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0$$

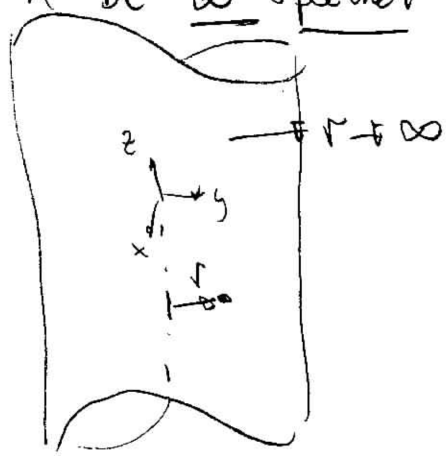


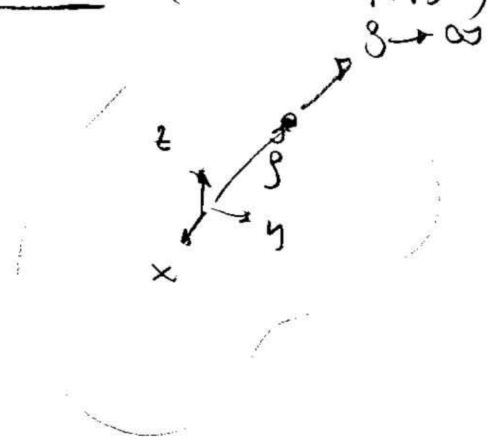
$T|_{\rho=0}$ = finite

$$\left.\frac{\partial T}{\partial \rho}\right|_{\rho=0} = 0$$

But these could be ∞ spatial dimension (do this first)







- First problem is...

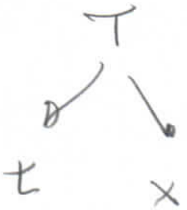
  Initially object is at uniform $T = T_i$ $(t=0)$

  Instantly change surface temp. to $T|_{surf} = T|_{r=0} = T_s$

  Go ∞ far out in material, eventually $T|_{r\to\infty} \to T_i$

"Non-dimensionalize" (sort of) $\Theta = T - T_i$ $\eta = \frac{x}{\sqrt{4\alpha t}}$ $\alpha = k/\rho c_p$

Convert $T(t,x)$ to $T(\eta) = T(\eta(t,x))$



We need to convert our PDE $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

Calc $\frac{\partial T}{\partial t} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial t} \overset{T.D.I}{=} \frac{-x}{2t\sqrt{4\alpha t}}\frac{dT}{d\eta}$

$\frac{\partial T}{\partial x} = \frac{dT}{d\eta}\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}}\frac{dT}{d\eta}$

keep going

$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta}\left(\frac{\partial T}{\partial x}\right)\frac{\partial \eta}{\partial x} = \frac{1}{4\alpha t}\frac{d^2 T}{d\eta^2}$

Put into PDE

Clean it up to $\boxed{\frac{d^2 T}{d\eta^2} = -2\eta \frac{dT}{d\eta}}$

Is now an ODE ... need 2 B.C.

Recall $\eta = x/\sqrt{4\alpha t}$

$T(t,x)$ problem

I.C. $t=0$; $T=T_i$

B.C. $x=0$; $T=T_s$

$x \to \infty$; $T \to T_i$

$T(\eta(t,x)) = T(\eta)$ problem

η B.C.

$\eta = 0$; $T = T_s$ (1)

$\eta \to \infty$; $T = T_i$ (2)

Don't dare ask about if $t=0$ <u>and</u> $x \to 0$!!!

We can solve this... $\dfrac{d^2T}{d\eta^2} = -2\eta\dfrac{dT}{d\eta}$

$$\frac{d\left(\frac{dT}{d\eta}\right)}{\left(\frac{dT}{d\eta}\right)} = -2\eta \quad \text{integrate...} \quad \ln\left(\frac{dT}{d\eta}\right) = -\eta^2 + C_0$$

$$\text{or } \frac{dT}{d\eta} = C_1 e^{-(\eta^2)}$$

Integrate again ...

$$T = C_1\int_{u=0}^{\eta} e^{-u^2}\,du + C_2$$

Apply B.C. (1)

$$T = C_1\cancel{\int_{\eta=0}^{\eta=0} e^{-u^2}}\,du + C_2 = T_s \qquad \boxed{C_2 = T_s}$$

0

Apply B.C. (2)

$$T\Big|_{\eta\to\infty} = T_i = C_1\int_{u=0}^{\infty} e^{-u^2}\,du + C_2$$

$$= C_1\frac{\sqrt{\pi}}{2} + T_s \qquad \boxed{C_1 = \frac{2(T_i - T_s)}{\sqrt{\pi}}}$$

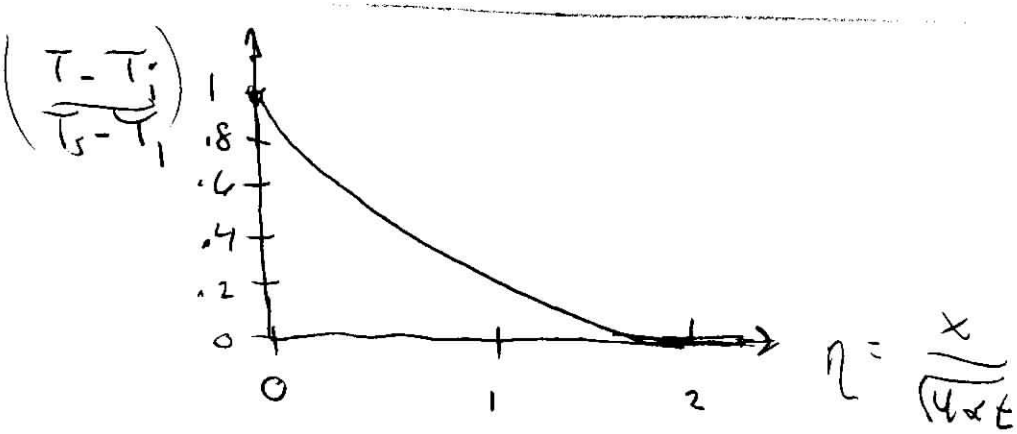
or... $$T = \underbrace{\frac{2(T_i - T_s)}{\sqrt{\pi}}}_{C_1} \int_{u=0}^{\eta} e^{-u^2}\,du + \underbrace{T_s}_{C_2}$$

$$\frac{T - T_s}{T_i - T_s} = \frac{2}{\sqrt{\pi}} \int_{u=0}^{\eta} e^{-u^2}\,du = \mathrm{erf}(\eta) = 1 - \mathrm{erfc}(\eta)$$

or

$$\frac{(T - T_i) + (T_i - T_s)}{T_i - T_s} = \frac{T - T_i}{T_i - T_s} + 1 = 1 - \mathrm{erfc}(\eta)$$

finally $$\boxed{\frac{T - T_i}{T_s - T_i} = \mathrm{erfc}(\eta) = \mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right)}$$



But we also want

$$q\Big|_{\eta=0} = q\Big|_{x=0} = -k\frac{dT}{dx}\Big|_{x=0} = -k\frac{dT}{d\eta}\frac{\partial \eta}{\partial x}\Big|_{\eta=0} = -k\,C_1 e^{-\eta^2}\frac{1}{\sqrt{4\alpha t}}\Big|_{\eta=0}$$

$$\boxed{q\Big|_{x=0} = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}}$$

Well, that was fun!

Lots o' work leads to

- Fixed heat flux at surface $q\big|_{x=0} = \text{¢} = q_s$

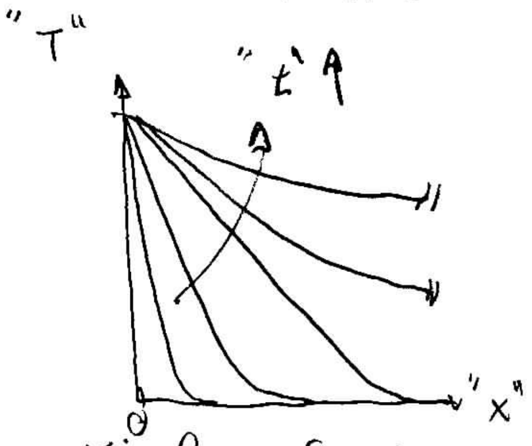
$$T(t,x) - T_i = \frac{q_s}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\,\exp\left(\frac{-x^2}{4\alpha t}\right) - x\cdot \operatorname{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right)\right]$$

Forge on...

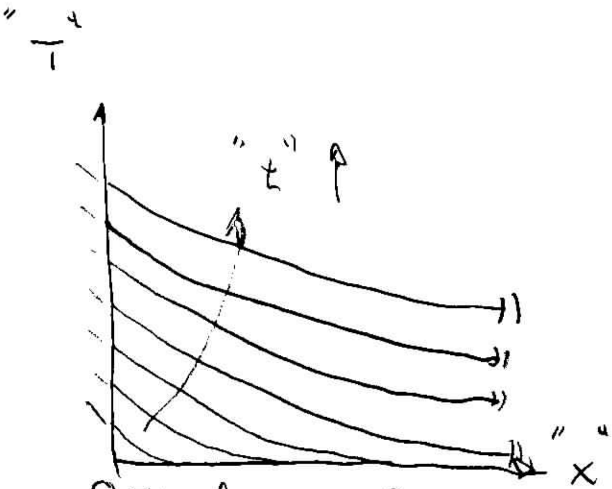
- Convection at surface $q\big|_{x=0} = q_s(t) = h\left[T_\infty - \underbrace{T(t,x=0)}_{T_s(t)}\right]$

$$\frac{T(t,x)-T_i}{T_\infty - T_i} = \operatorname{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right) - \exp\left(\frac{hx}{k} + \frac{h^2\alpha t}{k^2}\right)\operatorname{erfc}\left(\frac{x}{\sqrt{4\alpha t}} + \frac{h\sqrt{\alpha t}}{k}\right)$$

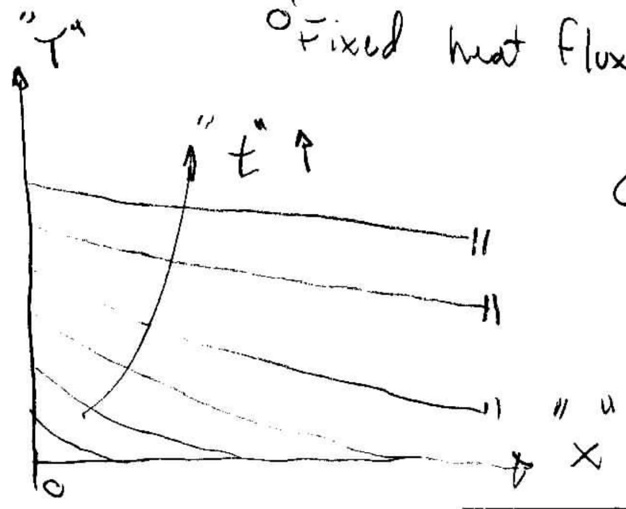
Plots "look like"



Fixed surface T_s

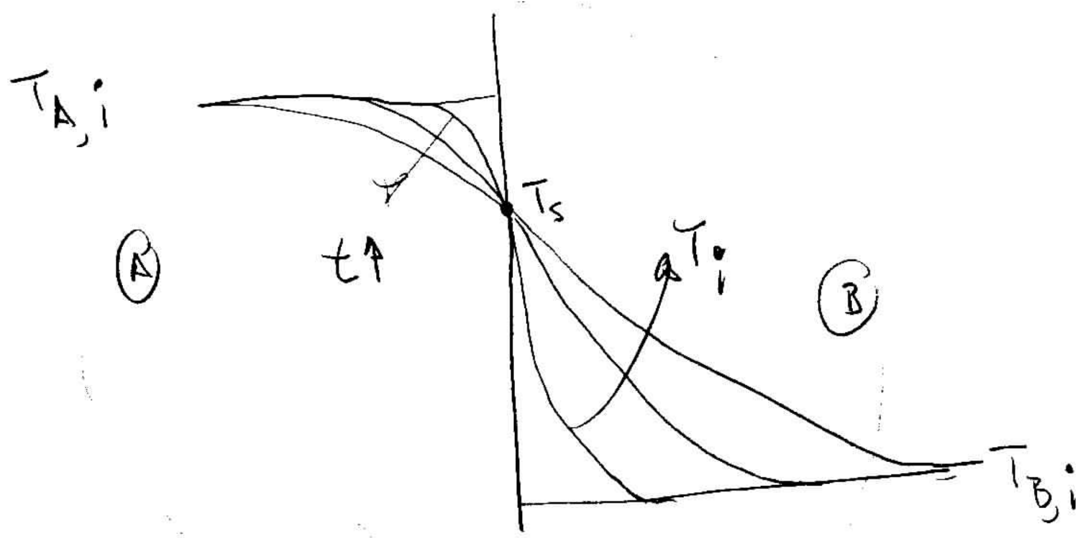


Fixed heat flux q_s



Conv. @ surface

• Contact of 2 walls of different materials will eventually be at the same temp $T = T_{\text{interface surface}} = T_s$



Each body looks like a semi-∞ wall problem with surface temperature T_s.

But

$$q_A = q_B$$

$$\frac{-k_A (T_s - T_{A,i})}{\sqrt{\pi \alpha_A t}} = \frac{-k_B (T_s - T_{B,i})}{\sqrt{\pi \alpha_B t}}$$

or

$$\frac{T_{A,i} - T_s}{T_s - T_{B,i}} = \frac{\sqrt{k \rho c_p}\,\}\,B}{\sqrt{k \rho c_p}\,\}\,A}$$ so that

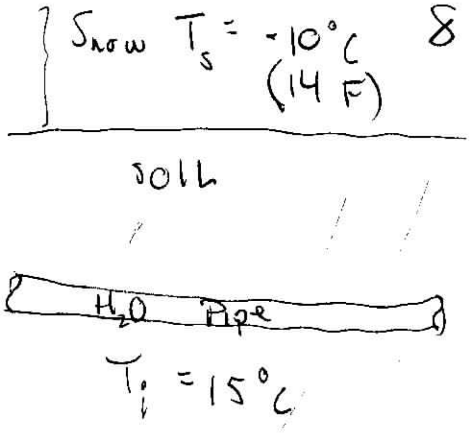
$$T_s = \frac{\sqrt{(k\rho c)_A}\, T_{A,i} + \sqrt{(k\rho c)_B}\, T_{B,i}}{\sqrt{(k\rho c)_A} + \sqrt{(k\rho c)_B}}$$

Think about touching wood vs. metal with your fingers
Which feels colder? why?

$T_{skin} \sim 35°C$
$T_{object} \sim 15°C$
$T_{s,Al} \sim 15.9°C$
wood $30°C$

$\sqrt{\ }_{Al} = 24 \frac{kJ}{m^2 K}$
$\sqrt{\ }_{wood} = 0.38$
$\sqrt{\ }_{skin} = 1.1$

Ex| How deep to bury H_2O main so it does not freeze?



- Snow cover lasts 3 months.
- $k_{soil} = 0.4 \frac{W}{mK}$
- $\alpha_{soil} = 0.15 \times 10^{-6} \ m^2/s$

from
$$\frac{T(t,x) - T_i}{T_s - T_i} = \operatorname{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right)$$

$$\frac{0 - 15}{-10 - 15} = \underbrace{\operatorname{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right)}_{\text{look at graph or a table}} = 0.60$$

$$\Rightarrow \eta \approx 0.37$$

so

$$x = \eta\sqrt{4\alpha t} = 0.80 \text{ m} \quad \approx 80 \text{ cm} \quad \approx 31.5 \text{ in}$$
$$2.62 \text{ ft}$$